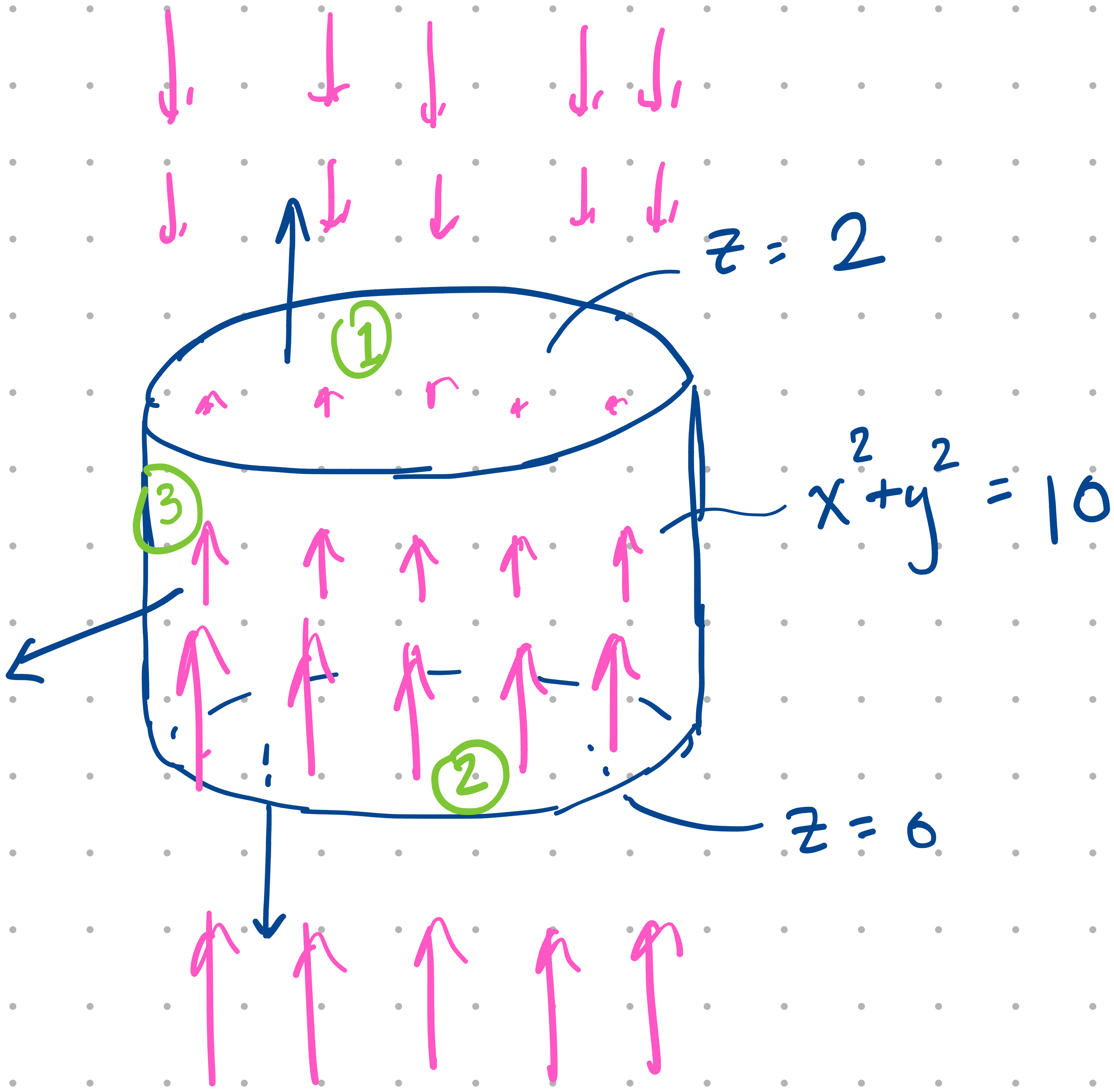

① Top:

$$\iint_{\text{Top}} \vec{F} \cdot d\vec{S} = 0$$

$\vec{F} = \vec{0}$ on the top face

② Bottom:

$$\iint_{\text{Bottom}} \vec{F} \cdot d\vec{S} = \iint_{\text{Bottom}} \langle 0,0,4 \rangle \cdot \langle 0,0,-1 \rangle \, dS$$

$$= \iint_{\text{Bottom}} -4 \, dS = -4 \, (\text{Area})$$

$$= \boxed{-40\pi}$$

③ Cylindrical side

$$\iint_{\text{Cylin.}} \underbrace{\vec{F} \cdot d\vec{S}}_{=0} = 0 .$$

b/c perpendicular

$\vec{F} = \langle 0, 0, 4 - z^2 \rangle$

$\vec{n} = \dfrac{\langle 2x, 2y, 0 \rangle}{|\langle 2x, 2y, 0 \rangle|}$

dot prod is zero.

A number of ways to produce $\hat{n}$ (unit normal):

① Sometimes geometrically obvious (i.e. a coordinate plane)

② If given a cartesian eq. $f(x,y,z) = 0$ describing surface:

$$\frac{\nabla f}{|\nabla f|}$$

③ If given a parametrization $\vec{r}(u,v)$:

$$\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

⚠ Make sure to check that the normal vector you compute agrees with the orientation you want on the surface!

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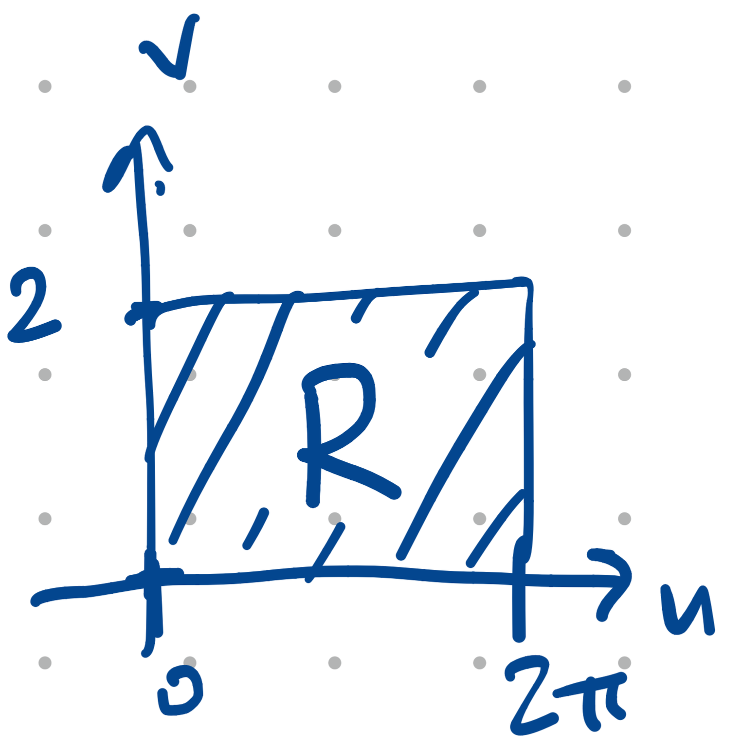
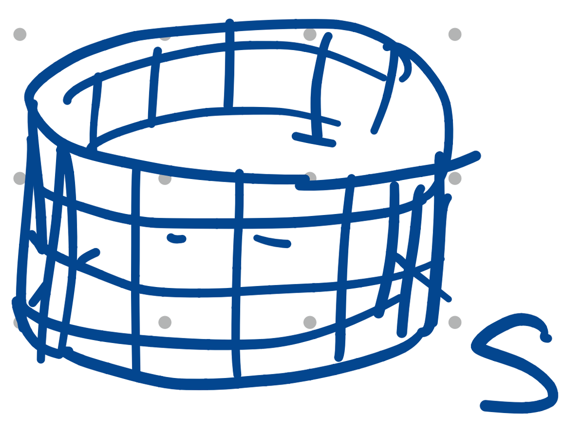
e.g. can parametrize cylindrical side of preceding problem:

$$\vec{r}(u,v) = \langle \sqrt{10}\cos u, \sqrt{10}\sin u, v \rangle$$





$0 \le u \le 2\pi$

$0 \le v \le 2$

Reasoning it out geometrically, see $\vec{r}_u \times \vec{r}_v$ is the way to produce outwards normal!

— OR —

$$\vec{r}_u(u,v) = \langle -\sqrt{10}\sin u, \sqrt{10}\cos u, 0 \rangle$$

$$\vec{r}_v(u,v) = \langle 0, 0, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle \sqrt{10}\cos u, \sqrt{10}\sin u, 0 \rangle$$

Then check some specific u,v to see if this is pointing in the right direction. (it is)

$$\iint \vec{F} \cdot d\vec{S}$$

$$= \iint \langle 0, 0, 4-v^2 \rangle \cdot \langle \sqrt{10}\cos u, \sqrt{10}\sin u, 0 \rangle \, du\, dv$$

$$= \int_0^2 \int_0^{2\pi} 0 \, du\, dv = 0.$$

Rmk:

$$\iint_{\text{surface}} \vec{F}\cdot \underbrace{\vec{n}\, dS}_{d\vec{S}} = \iint_{\substack{u,v \\ \text{region}}} \vec{F}\cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cancel{|\vec{r}_u \times \vec{r}_v|}\, du\, dv$$

i.e. no need to divide by $|\vec{r}_u \times \vec{r}_v|$ when doing flux integrals by param. b/c it just cancels.

Analogous to

$$\int_{\text{curve}} \vec{F}\cdot \underbrace{\vec{T}\, ds}_{d\vec{r}} = \int_{t_0}^{t_1} \vec{F}\cdot \frac{\vec{r}'(t)}{\cancel{|\vec{r}'(t)|}} \cancel{|\vec{r}'(t)|}\, dt$$

A problem related to example from last time:

$$\iint_D \langle x,y,z\rangle \cdot d\vec{S}$$

where D is $x^2+y^2+z^2=9$ oriented outwards.

(Hint: surface area of sphere of radius R is $4\pi R^2$)

Sol: $\vec{n} = \dfrac{\langle 2x, 2y, 2z\rangle}{6} = \langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3}\rangle$

b/c $\sqrt{4x^2+4y^2+4z^2} = 6$

$$\iint_D \langle x,y,z\rangle \cdot \langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3}\rangle \, dS$$

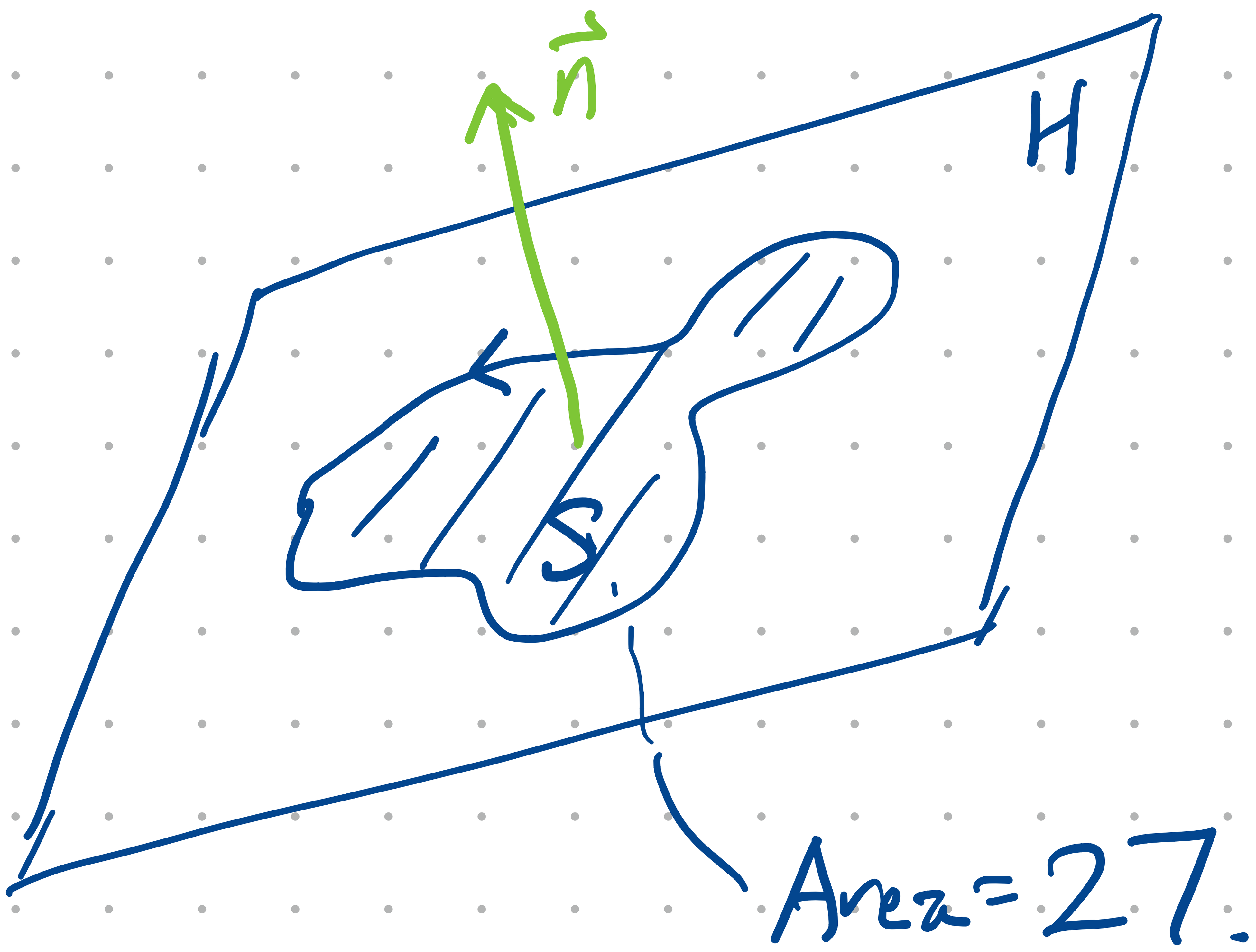
$$= \iint_D \frac{x^2+y^2+z^2}{3} \, dS = \iint_D 3 \, dS$$

$$= 3(4\pi 9) = \boxed{108\pi}$$

Problem: Let H be the plane $3x+4y+z=7$.

Let $\vec{F} = \langle 0, x, 0 \rangle$

Let C be a closed curve on the plane H,



CCW when viewed from above

Question: What is $\oint_C \vec{F} \cdot d\vec{r}$?

Sol:

$$\dots = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_S \langle 0, 0, 1 \rangle \cdot d\vec{S}$$

$$= \iint_S \underbrace{\langle 0,0,1\rangle \cdot \frac{\langle 3,4,1\rangle}{|\langle 3,4,1\rangle|}}_{\text{constant } C \text{ you can evaluate}} dS$$

positive z, so this is in right direction

$$= 2\pi C.$$